Suddhendu Biswas and Nather Abas Ebraheem

Applicability of the Malthusian Law of Population Growth : A Note on India's Experience*

1. Introduction

THE accelerating population growth rates of India have lead to a belief among a section of demographers that Malthusian Law of population growth is valid for explaining the phenomenally high growth rates over decades. The object of the present paper is to test the hypothesis in the light of the Quasi-Stable population theory, based on the assumption of consistently decreasing fertility rates corresponded by a negligible change in the mortality pattern which is believed to be a pertinent model for the dynamics of India's population by many population theorists.

It is known that a Quasi-Stable population with stable fertility component and constantly decreasing mortality trend eventually gives rise to accelerating growth rates. This paper, therefore, examines whether such accelerating growth rates in the context of moderately changing fertility rates can give rise to a geometric pattern of growth rate at least for a few decades. This may lend credence to a hypothesis revalidating Malthusian theory.

2. Assumptions and the Methodology

Let $i(x, t)$ and $p(x, t)$ denote the probability of giving birth between the age $(x, x + \Delta x)$ as $\Delta x \rightarrow 0$ at time t and $p(x, t)$ stands for the survival probability

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between the age $(x, x + \Delta x)$ as $\Delta x \to 0$ at time (t) then Coale's (1972) assumption for the Quasi-Stable population with decreasing fertility and constant mortality is given by

$$i(x, t) = e^{kt}\, i(x, 0) = e^{kt}\, i(x) \quad \text{for} \quad k < 0. \tag{0}$$

Generalizing the same for our defined population, characterized by decreasing fertility and increasing survival probability subject to the condition that decrease in fertility is more than that of increase in the survival probability we have the following :

$$\text{(i)}\ i(x, t) = i^{k_1 t}\,(x, 0) \qquad \text{where } k_1 = k_1(t)$$
$$= e^{k_1 t}\,(i(x), \qquad k_1 < 0;\ \alpha \leqslant x \leqslant \beta \tag{A}$$

$$\text{(ii)}\ p(x, t) = e^{k_2 t}\, p(x, 0) \qquad \text{where } k_2 = k_2(t)$$
$$= e^{k_2 t}\, p(x), \qquad k_2 > 0;\ \alpha \leqslant x \leqslant \beta \tag{B}$$

(iii) $R(t)$ = Net Reproduction Rate (N.R.R.) per women per year.

(iv) $B(t)$ = # Births at time (t)

(v) (α, β) being the upper and lower age limits of child bearing interval.

Then,

$$\int_\alpha^\beta p(x, t)\, i(x, t)\, dx = e^{(k_1+k_2)t} \int_\alpha^\beta i(x)\, p(x)\, dx \tag{2.1}$$
$$= e^{(k_1+k_2)t}\, R(0)$$

where $R(0)$ is the Net reproduction rate (N.R.R.) at $t = 0$.

Also :

$$B(t) = \int_\alpha^\beta B(t - x)\, p(x, t)\, i(x, t)\, dx$$
$$= \left[\int_\alpha^\beta B(t - x)\, p(x)\, i(x)\, dx \right] e^{(k_1+k_2)t}$$
$$= B(t - T(t)) \left[\int_\alpha^\beta p(x)\, i(x)\, dx \right] e^{(k_1+k_2)t}$$ using (2.1) by the Mean Value Theorem of Integral calculus where $\alpha < T(t) < \beta$

The difference equation in $B(t)$ given by

$$B(t) = R(0)\, e^{(k_1 + k_2)t} B(t - T(t)) \tag{2.2}$$

Following the technique of the improved solution of (2.2) by Coale, by employing the second method and generalizing the assumption of Coale by (A) and (B) we get the first approximate parabolic solution of (2.2) by routine calculation given by

$$B(t) = B(0)\, e^{\frac{(k_1+k_2)}{2} t + \frac{(k_1+k_2)}{2T_0} t^2} \tag{2.3}$$

(for proof vide reference (2))

where T_0 is the mean age of child bearing (approximately taken to be equal to $T_{(t)}$ by the assumption of Coale's first method). After some adjustments following coale's (1972) approach of the second method following structure of $B(t)$ (again with routine calculation) the improved solution of (2.2) satisfying

$$B(t) = R(0)\, e^{(k_1+k_2)t}\, B(t - T_0) \left[e^{-\sigma^2 \frac{(k_1+k_2)^2}{4T_0} t + \frac{(k_1+k_2)^2}{2T_0^2} \sigma^2 t^2 + \frac{(k_1+k_2)}{2T_0} \sigma^2} \right]$$

(for proof vide appendix (A)) (2.4)

where σ^2 is the variance of the corresponding stable age distribution denoting by $Y(t) = \log_e B(t)$ and putting $R(0) = 1$, we have

$$Y(t) - Y(t - T_0) = (k_1 + k_2)\, t - \frac{\sigma^2 (k_1 + k_2)^2}{4T_0}\, t + \frac{(k_1 + k_2)^2}{2T_0^2}\, \sigma^2 t^2 + \frac{(k_1 + k_2)}{2T_0}\, \sigma^2$$

The solution of the difference equation (2.2) is therefore given by :

$$B(t) = B(0)\, e^{\left[\frac{(k_1+k_2)}{2} + \sigma^2 \frac{(k_1+k_2)}{2T_0} \left[\frac{1}{T_0} - \frac{(k_1+k_2)}{12} \right] t + \frac{(k_1+k_2)}{2T_0} \left[1 + \frac{\sigma^2 (k_1+k_2)}{4T_0} \right] t^2 + \frac{\sigma^2 (k_1+k_2)^2}{6T_0^3} t^3 \right]} \tag{2.5}$$

(for proof vide appendix (A))

Now we have,

$P(t) = \int_0^{\omega} B(t - x)\, p(x, t)\, dx$, where ω is the highest possible age for survival.

Putting the parabolic solution (2.3) for $B(t - x)$ we have

$$P(t) = B(0) \int_0^{\omega} e^{\frac{(k_1+k_2)}{2}(t - x) + \frac{(k_1+k_2)}{2T_0}(t - x)^2}\, p(x, t)\, dx$$

$$= B(t) \int_0^{\omega} e^{-\frac{(k_1+k_2)}{2} x + \frac{(k_1+k_2)}{2T_0} x^2 - \frac{(k_1+k_2)}{T_0} tx}\, p(x, t)\, dx \tag{2.6}$$

$$\Rightarrow P(t+1) = B(t+1) \int_0^{\omega} e^{-\frac{(k_1+k_2)}{2} x + \frac{(k_1+k_2)}{2T_0} x^2 - \frac{(k_1+k_2)}{T_0} (t+1)x} p(x, t+1)\, dx$$

$$= B(t+1) \int_0^{\omega} e^{-\frac{(k_1+k_2)}{2} x + \frac{(k_1+k_2)}{2T_0} x^2 - \frac{(k_1+k_2)}{T_0} tx} e^{-\frac{(k_1+k_2)}{T_0} x} p(x, t) \left[\frac{p(x, t+1)}{p(x, t)}\right] dx \qquad (2.7)$$

where '10' year period is taken as the unit of time. By the mean value theorem of integral calculus we have,

$$\left[\frac{P(t+1)}{P(t)}\right] = \left[\frac{B(t+1)}{B(t)}\right]\left[\frac{p(\pi, t+1)}{p(\pi, t)}\right]\left[e^{-\frac{(k_1+k_2)}{T_0}\pi}\right] \qquad (2.7')$$

$$0 \leqslant \pi \leqslant \omega$$

Putting $\frac{(k_1+k_2)}{T_0} = -k' = -k'(t)$ and

$$C(t) = \left[\frac{B(t+1)}{B(t)} \frac{p(\pi, t+1)}{p(\pi, t)}\right] \quad k', C > 0$$

we have

$$\left[\frac{P(t+1)}{P(t)}\right] = C(t)\, e^{k'(t)\pi} \qquad (2.8)$$

(2.8) $\Rightarrow$

$$P(t+1) = C(t)\, e^{k'(t)\pi}\, P(t)$$
$$P(t+2) = C(t+1)\, e^{k'(t+1)\pi}\, P(t+1)$$
$$\Rightarrow P(t+2) = C(t+1)\, C(t)\, e^{k'(t+1)\pi}\, e^{k'(t)\pi}\, P(t)$$
$$P(t+n) = [C(t+n-1)\, C(t+n-2) \ldots C(t)], [e^{k'(t+n-1)\pi}\, e^{k'(t+n-2)\pi} \ldots e^{k'(t)\pi}]\, P(t)$$

or

$$P(t+n) = \left[\prod_{i=1}^{n} C(t+n-i)\, re^{k'(t+n-i)\pi}\right] P(t) \qquad (2.8')$$

which shows that the population size will be continuously increasing provided each of $k'(t)$'s are now negative and $C(t)$'s are greater than unities. This necessitates

$$\frac{B(\tau+1)}{B(\tau)} > \frac{p(\pi, \tau)}{p(\pi, \tau+1)} \quad \text{and } k'(\tau) > 0 \text{ for } \tau = t, t+1, t+2 \ldots$$

Also

$$\frac{P(\tau + 1)}{P(\tau)} = C(\tau)\, e^{k'(\tau)\pi},\ \tau = t,\ t + 1,\ t + 2, \ldots$$

holding for two consecutive decennium, may be applied over individual years in the following pattern :

$$P\left(\tau + \frac{1}{10}\right) = P(\tau)\, [C(\tau)\, e^{k'(\tau)\pi}]^{1/10}$$

Similarly,

$$P\left(\tau + \frac{2}{10}\right) = P(\tau)\, [C(\tau)\, e^{k'(\tau)\pi}]^{2/10}$$

and

$$P\left(\tau + \frac{x}{10}\right) = P(\tau)\, [C(\tau)\, e^{k'(\tau)\pi}]^{x/10} \qquad 0 \leqslant x \leqslant 10$$

So that the inter decinnal growth rate between τ to $(\tau + 1)$ may be assumed to take place in a geometric rate with common ratio

$$[C(\tau)\, e^{k'(\tau)\pi}]^{1/10} \quad \text{for} \quad \tau = t,\ t + 1,\ t + 2, \ldots .$$

$C(\tau)$ and $k'(\tau) > 0$ but monotonically decreasing overtime (as the empirical part of the exercise shows), whereas the growth rate over decades will be continuously decreasing leading to the convergence to stability.

Hence as long as $\dfrac{B(\tau + 1)}{B(\tau)} > \dfrac{p(\pi, \tau)}{p(\pi, \tau + 1)}$, $k'(\tau) > 0$

One may have apparently a picture of the geometric rate of population increase within decennium nevertheless the growth rate has a decreasing pattern if comparison is made over decades which is a characteristic of the population converging towards stability.

3. Estimation Technique and the Basic Data Employed

Let $P(\tau)$ represent the female population corresponding to the year τ and,

$$B(\tau) = \text{female birth during } \tau \cdot \tau = t,\ t + 1,\ t + 2, \ldots$$

where t, $t + 1$, $t + 2$, $t + 3$ stand for 1971, 1981, 1991 and 2001 A.D. respectivey.

Then, we start estimating the parameters $(k_1 + k_2)$ and T_0 by using the ratio of the population age distribution to that of a corresponding stable population having the same demographic parameters of fertility and mortality as given by

$$\frac{C(x,\tau)}{C_s(x,\tau)} = e^{k_2\tau}\,\frac{b(\tau)}{b_s(\tau)}\,e^{-\frac{(k_1(\tau)+k_2(\tau))}{2}x+\frac{(k_1(\tau)+k_2(\tau))}{2T_0}x^2}$$

$$\text{for } \tau = t,\ t+1 \qquad (3.1)$$

(for proof vide appendix (B))

Taking logarithm and employing the method of summation in (3.1) we have

$$\sum \log_e \frac{C(x,\tau)}{C_s(x,\tau)} = n\log_e\left[e^{k_2(\tau)}\frac{b(\tau)}{b_s(\tau)}\right] - \frac{(k_1(\tau)+k_2(\tau))}{2}\sum x + \frac{(k_1(\tau)+k_2(\tau))}{2T_0}\sum x^2 \qquad (3.2)$$

$$\sum x\log_e \frac{C(x,\tau)}{C_s(x,\tau)} = \log_e\left[e^{k_2(\tau)}\frac{b(\tau)}{b_s(\tau)}\right]\sum x - \frac{[(k_1(\tau)+k_2(\tau)]}{2}\sum x^2 + \frac{[k_1(\tau)+k_2(\tau)]}{2T_0}\sum x^3 \qquad (3.3)$$

$$\sum x^2\log_e \frac{C(x,\tau)}{C_s(x,\tau)} = \log_e\left[e^{k_2(\tau)}\frac{b(\tau)}{b_s(\tau)}\right]\sum x^2 - \frac{[k_1(\tau)+k_2(\tau)]}{2}\sum x^3 + \frac{[k_1(\tau)+k_2(\tau)]}{2T_0}\sum x^4 \qquad (3.4)$$

Having known the values of $C(x, \tau)$ and $C_s(x, \tau)$ from the sources (S_1) and (S_2) as given in the later part of the same section for $\tau = t,\ t+1$, the three estimating equations (3.2), (3.3) and (3.4) were solved to find out $(k_1 + k_2)$, T_0 from the three unknowns

$$\left[e^{k_2(\tau)}\frac{b(\tau)}{b_s(\tau)}\right], \left[\frac{k_1(\tau)+k_2(\tau)}{2}\right] \text{ and } \left[\frac{k_1(\tau)+k_2(\tau)}{2T_0}\right] \text{for } \tau = t,\ t+1.$$

In the next place after having estimated $(k_1 + k_2)$ and T_0 for $\tau = t,\ l + 1$ we substituted the estimate for $\tau = t,\ t + 1$ in

$$\log_e\left[\frac{P(\tau+1)}{P(\tau)}\ \frac{B(\tau)}{B(\tau+1)}\right] = \log_e\left[\frac{p(\pi,\tau+1)}{p(\pi,\tau)}\right] - \left[\frac{[k_1(\tau)+k_2(\tau)]}{T_0}\right]\pi \qquad (3.5)$$

(which is obtainable by taking logarithms on both sides of (2.7′), π is directly estimated (when $p(\tau + 1)$, $p(\tau)$ and $(B(\tau + 1)$ and $B(\tau))$ are known from sources (S_1), (S_2) and (S_3) (vide later part of its section) respectively), subject to the assumption of $[p(\pi, \tau + 1)/p(\pi, \tau)]$ as given in (S_4).

Finally, while substituting the estimates for $(k_1 + k_2)$, T_0 and π for $\tau = t + 1$ and making the assumptions of (S_3) and (S_4) relating to $[p(\pi, \tau + 1)/p(\pi, \tau)]$ and $[B(\tau)/B(\tau + 1)]$ for $\tau = (t + 1), (t + 2)$ respectively in (3.5) we get the population estimates of $p(t + 2)$ (i e. 1991 A.D.) and $P(t + 3)$ (2001 A.D.). The following sources of data were utilized to obtain the estimates :

(S_1) : *Census of India 1981, Series I 'Age Tables, Based on 5% Sample Data,* for the age distribution of the female population (denoted by $C(x, \tau)$ for all x and $\tau = t + 1$ (i.e. 1981 A.D.) as well as the total female population $P(\tau)$ $(\tau = t + 1)$ (Table pp. 35). This gives $P(t + 1) = 321357426$ as the figure for the mid year female population during the decade 1971-1981.

The mid year female population figure of 1961-1971 A.D. (i.e. $P(t)$ is obtainable from source (S_3) as given below.

(S_2) : *U.N. 1957 'Methods of estimating basic demographic measures from incomplete data'* for getting the expectation of life at birth for the comparable stable population (possessing the same Demographic parameters of fertility and mortality) with that of the female population of India 1981 A.D. and the corresponding stable age distribution $C_s(x, \tau)$ for $\tau = t + 1$, $C_s(x, t)$ is obtained by interpolating between levels 13 and 15 as in the corresponding tables.

(S_3) : The female births in the decade 1961-1970 and 1971-1980, denoted by $B(t)$ and $B(t + 1)$ were obtained from *Census of India 1971, Series I paper 3 (1977)' Age tables* for the total female population for 1971. This gives the estimates of the female births in 1971 and 1981 A.D. as follows :

$$B(t) = 4653977 \quad \text{for} \quad 1961\text{-}1970 \tag{3.6}$$

$$B(t + 1) = 5272635 \quad \text{for} \quad 1971\text{-}1980 \tag{3.7}$$

and $$P(t) = 264110376 \quad \text{for} \quad 1961\text{-}1970 \tag{3.8}$$

(S_4) : We assume the ratios of the survival probabilities $\tau = t, t + 1, t + 2, t + 3$ for the year τ to $(\tau + 1)$ at the age π $(0 \leqslant \pi \leqslant \omega)$ in increasing order as given by as per our basic assumption (B).

$$\left[\frac{p(\pi, t + 1)}{p(\pi, t)}\right] = 1.005, \left[\frac{p(\pi, t + 2)}{p(\pi, t + 1)}\right] = 1.006$$

and $$\left[\frac{p(\pi, t + 3)}{p(\pi, t + 2)}\right] = 1.007$$

(S_5) Also per our assumption in (A) we take

$$\left[\frac{B(\tau + 2)}{B(\tau + 1)}\right] = (0.99)\left[\frac{B(\tau + 1)}{B(\tau)}\right] \quad \text{for} \quad \tau = t, t + 1,$$

substituting the values of $P(t+1)$, $P(t)$, $B(t+1)$, $B(t)$, and $(k_1 \hat{+} k_2)|_{t+1} = -.0100684$, $\hat{T}_0 = 29$ and $p(\pi, t+1)/p(\pi, t)$ in (3.5) obtain the estimate of $\hat{\pi} = 38$ for the year 1981 A.D.

Using (S_4, S_5) we get from equation (3.5) for $\tau = t+1, t+2$

$$\frac{P(t+2)}{P(t+1)} = 1.1947007 \tag{3.9}$$

and

$$\frac{P(t+3)}{P(t+2)} = 1.1845276 \tag{3.10}$$

Using the figure of $P(t+1)$ for 1981 given in (S_1) we get

$$P(t \hat{+} 2)|_{1991} = 387464000 \tag{3.11}$$

and using the estimated value of $P(t+2)|_{1991}$ we get

$$P(t \hat{+} 3)|_{2001} = 462963000 \tag{3.12}$$

Finally by using (B)

$$p(x, t) = p(x, 0)\, e^{k_2 t}$$

it is possible to estimate k_1 and k_2 separately from the already obtained estimates of $(k_1 + k_2)$ corresponding to $t = 1971$ and $(t+1) = 1981$ A.D. respectively as follows :

Putting $x = \pi$ in (B) and replacing t by $t+1$ in (B) we have

$$p(\pi, t+1) = p(\pi, t)\, e^{k_2 t}$$

$$\Rightarrow \quad \frac{p(\pi, t+1)}{p(\pi, t)} = e^{k_2 t}$$

$\hat{k}_2|_t = .00498$ where $\hat{k}_2|_t$ denotes the estimate of k_2 at time t.

Similarly by using $\dfrac{p(\pi, t+2)}{p(\pi, t+1)} = 1.006$ and $\dfrac{p(\pi, t+3)}{p(\pi, t+2)} = 1.007$ we have

$$\left[\hat{k}_2\Big|_{t+1}\right] = .00299$$

$$\left[\hat{k}_2\Big|_{t+2}\right] = .00232$$

$$\left[\hat{k}_2\Big|_{t+3}\right] = .00146$$

Having already estimated $(k_1 + k_2)$ for t, $t + 1$ the estimates of $(k_1 + k_2)$ for the years $(t + 2)$ and $(t + 3)$ making the following assumptions viz.

$$\frac{(k_1 + k_2)|_{t+1}}{(k_1 + k_2)|_t} \simeq \frac{(k_1 + k_2)|_{t+2}}{(k_1 + k_2)|_{t+1}} \simeq \frac{(k_1 + k_2)|_{t+3}}{(k_1 + k_2)|_{t+2}}$$

This gives the individual estimates of k_1 and k_2 at different time periods given in Table 1.

TABLE 1—ESTIMATES OF k_1 AND k_2 AT DIFFERENT TIME PERIODS

Year	$\hat{k}_1$	$\hat{k}_2$	$(k_1 \hat{+} k_2)$
1971	−.0743	.00498	−.0693165
1981	−.013058	.00299	−.0100684
1991	−.010925	.00232	−.0086
2001	−.00881	.00146	−.00735

4. Conclusion

The foregoing calculation has shown that :

$$\left[\frac{B(t+1)}{B(t)}\right] = 1.132931 \tag{4.1}$$

$$\left[\frac{B(t+2)}{B(t+1)}\right] = 1.1216016 \tag{4.2}$$

$$\left[\frac{B(t+3)}{B(t+2)}\right] = 1.1103855 \tag{4.3}$$

and with the assumptions of

$$\left[\frac{p(\pi, t)}{p(\pi, t+1)}\right] = .9950248 \tag{4,4}$$

$$\left[\frac{p(\pi, t+1)}{p(\pi, t+2)}\right] = .9940357 \tag{4.5}$$

$$\left[\frac{p(\pi, t+2)}{p(\pi, t+3)}\right] = .9930986 \tag{4.6}$$

$$\Rightarrow \left[\frac{B(t+1)}{B(t)} \; \frac{p(\pi, t+1)}{p(\pi, t)}\right] = C(t) = 1.1375906 \tag{4.7}$$

$$\Rightarrow \left[\frac{B(t+2)}{B(t+1)}\,\frac{p(\pi, t+2)}{p(\pi, t+1)}\right] = C(t+1) = 1.1283312 \tag{4.8}$$

$$\Rightarrow \left[\frac{B(t+3)}{B(t+2)}\,\frac{p(\pi, t+3)}{p(\pi, t+2)}\right] = C(t+2) = 1.1181518 \tag{4.9}$$

Since the condition

$$\frac{B(\tau+1)}{B(\tau)} > \frac{p(\pi, \tau)}{p(\pi, \tau+1)} \quad \text{and} \quad k'(\tau) > 0.$$

holds for all periods $\tau = t, t+1, t+2$ (i.e.) from 1971 to 1991 A.D. it follows that, although the population may show an apparently geometric rate of increase within the decennium but ultimately the growth rate decreases leading the population towards stability which annihilates the apprehension of a possible Malthusian rate of growth rate in India. As exhibited in this exercise is :

$$\frac{1}{1000}\left[\frac{387464000 - 321357466}{321357466}\right] \times 100 = 1.95\% \text{ per annum per 1000}$$

population between 1981-1991 A.D. and that between 1991 A.D. to 2001 A.D. the growth rate will be

$$\frac{1}{1000}\left[\frac{462963000 - 387464000}{387464000}\right] \times 100 = 1.84\% \text{ per annum per 1000 population.}$$

Also the growth rate when calculated back for the 1971-1980 female population comes out to be 2.17% ($\simeq$ 2.2%) which compares favourably with the actual growth rate for male and female population combined as 2.5%* (unadjusted for migration). The conclusion in respect of the testing whether Indian population growth follows the Malthusian law has been made in respect of female population. However, the conclusion for male populations on the basis of projected Male populations as per the model will lead to same conclusion on the assumption of the sex ratio at different age groups remaining invariant during the periods under consideration.

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*Source : vide refer 6 (table 2).

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APPENDIX A

We have

$$B(t) = R(0)\, e^{(k_1+k_2)t)}\, B(t - T_{(t)}) \tag{A}$$

To have an improved solution, we write

$$B(t) = R(0)\, e^{(k_1+k_2)t}\, \frac{B^*(t - T_{(t)})}{B^*(t - T_0)}\, B(t - T_0) \tag{B}$$

if $B^* = B$ then (A) and (B) becomes the same then;

$$B^*(t) = B(0)\, e^{\frac{(k_1 + k_2)}{2} t + \frac{(k_1 + k_2)}{2T_0} t^2} \tag{C}$$

and we put $T_{(t)} = T$ (Mean length of generation in stable population) taking $\log_e$ we have

$$\log_e B^* \frac{(t - T_{(t)})}{B^*(t - T_0)} = \log_e B^*(t - T) - \log_e B^*(t - T_0) \tag{D}$$

by Taylor's expantion, we have

$$\log_e B^*(t - T) = \log_e B^*(t - T_0) + (T_0 - T)\, \frac{\delta \log_e B^*(t)}{\delta t}\bigg|_{t=t-T_0} \tag{E}$$

Taking $\log_e$ derivative on both sides of (c)

$$\frac{d \log_e B^*(t)}{dt} = \frac{d}{dt}\left[\log_e B(0) + \frac{(k_1 + k_2)}{2}\, t + \frac{(k_1 + k_2)}{2T_0}\, t^2\right]$$

$$\Rightarrow \frac{\delta \log_e B^*(t)}{\delta t}\bigg|_{t=t-T_0} = \frac{(k_1 + k_2)}{2}\bigg|_{t=t-T_0} + \frac{(k_1 + k_2)}{2T_0}\, 2t\bigg|_{t=t-T_0}$$

$$= \frac{(k_1 + k_2)}{2} + \frac{(k_1 + k_2)}{T_0}\, (t - T_0)$$

$$= \frac{(k_1 + k_2)}{T_0}\, t - \frac{(k_1 + k_2)}{2}$$

Substituting in (E) we obtain

$$\log_e \frac{B^*(t - T)}{B^*(t - T_0)} = (T_0 - T)\left[\frac{(k_1 + k_2)}{T_0}\, t - \frac{(k_1 + k_2)}{2}\right] \tag{F}$$

Although the approximation viz;

$$\frac{B^*(t - T)}{B^*(t - T_0)} \simeq \frac{B(t - T)}{B(t - T_0)}$$ is justified.

But,

$$B^*(t) = B(0)\, e^{\frac{(k_1 + k_2)}{2} t + \frac{(k_1 + k_2)}{2T_0} t^2}$$

is a parabolic solution and it will not be same as $B(t)$ of the stable population which depends on the level of fertility. Again, we have,

$$B^*(t) = \int_\alpha^\beta B^*(t - x)\, i(x, t)\, p(x, t)\, dx$$

$$= \int_\alpha^\beta B(0)\, e^{\frac{(k_1 + k_2)}{2} (t - x) + \frac{(k_1 + k_2)}{2T_0} (t - x)^2}\, i(x, t)\, p(x, t)\, dx$$

(by replacing t by $(t - x)$ in both sides of (C))

$$B^*(t) = B(0)\, e^{\frac{(k_1 + k_2)}{2} t + \frac{(k_1 + k_2)}{2T_0} t^2} \int_\alpha^\beta [e^{\frac{(k_1 + k_2)}{2} x + \frac{(k_1 + k_2)}{2T_0} x^2}$$

$$\cdot\, e^{\frac{(k_1 + k_2)}{T_0}]xt}\, i(x, t)\, p(x, t)]\, dx$$

$$B^*(t) = B^*(t) \int_\alpha^\beta e^{-\frac{(k_1 + k_2)}{2} x + \frac{(k_1 + k_2)}{2T_0} x^2 - \frac{(k_1 + k_2)}{T_0} xt}\, i(x, t)\, p(x, t) dx$$

$$1 = \int_\alpha^\beta e^{-\frac{(k_1 + k_2)}{2} x + \frac{(k_1 + k_2)}{2T_0} x^2 - \frac{(k_1 + k_2)}{T_0} tx}\, i(x, t)\, p(x, t)\, dx$$

Replacing $T_0 = T$ and $e^{-\frac{(k_1 + k_2)}{T_0} tx} = e^{-r(t)x}$ Hence

$$\int_\alpha^\beta e^{-\frac{(k_1 + k_2)}{2} x + \frac{(k_1 + k_2)}{2T} x^2 - r(t)\, x}\, i(x, t)\, p(x, t)\, dx = 1 \qquad \text{(G)}$$

it differs from Lotka's integral equation unless $(k_1 + k_2) = 0$. We therefore consider the behaviour of the R.H.S. of (G) for non-zero $(k_1 + k_2)$. Let

$$Z(t) = \int_\alpha^\beta e^{-\frac{(k_1 + k_2)}{2} x + \frac{(k_1 + k_2)}{2T_{(t)}} x^2 - r(t)\, x}\, p(x, t)\, i(x, t)\, dx$$

The idea is to express $Z(t)$ in terms $Z(t)\,|_{k_1+k_2=0}$ and therefore incorporates the correction necessary in $B^*(t)$ enabling to satisfy the Lotka's integral equation we have

$$Z(t) \simeq Z(t)\,|_{(k_1+k_2)=0} + (k_1 + k_2)\,\frac{dZ}{d(k_1 + k_2)}\,(k_1 + k_2) = 0 + 0(k_1 + k_2)$$

$$\frac{dZ(t)}{d(k_1 + k_2)} = \int_{\alpha}^{\beta} e^{-\frac{(k_1 + k_2)}{2}x + \frac{(k_1 + k_2)}{2T_{(t)}}x^2}\, e^{-r(t)x}\, p(x, t)\, i(x, t)\, dx$$

$$\left[-\frac{x}{2} + \frac{x^2}{2T_{(t)}}\right]$$

$$\left.\frac{dZ(t)}{d(k_1 + k_2)}\right|_{(k_1+k_2)=0} = -\frac{1}{2}\int_{\alpha}^{\beta} xe^{-r(t)x}\, p(x, t)\, i(x, t)\, dx$$

$$+ \frac{1}{2T_{(t)}}\int_{\alpha}^{\beta} x^2\, e^{-r(t)x}\, p(x, t)\, i(x, t)\, dx$$

$$= -\frac{1}{2}\Lambda + \frac{1}{2T_{(t)}}\,[Y_2 + \Lambda^2]$$

where Y_2 is the variance of the age of the net fertility schedule in the stable population and Λ is the mean age of the net fertility schedule in the stable population, if we assume $\Lambda = T_{(t)}$

$$\left.\frac{dZ(t)}{dt}\right|_{(k_1+k_2)=0} = -\frac{1}{2}\Lambda + \frac{1}{2\Lambda}\,[Y_2 + \Lambda^2]$$

$$= -\frac{1}{2\Lambda}\,Y_2 \cong \frac{\sigma^2}{2T_0} \text{ where } Y = \sigma^2 \text{ and } T = T_0$$

$$Z(t) = Z(t)\,|_{(k_1+k_2)=0} + (k_1 + k_2)\left.\frac{dZ}{dt}\right|_{(k_1+k_2)=0}$$

$$= 1 + (k_1 + k_2)\frac{\sigma^2}{2T} \cong e^{(k_1+k_2)}\,\frac{\sigma^2}{2T} \qquad \text{(H)}$$

we have

$$\int_{\alpha}^{\beta} B(T - x)\, f(x)\, dx = B(T - T_{(t)})$$

where

$$f(x, t) = \frac{p(x)\, i(x)\, e^{(k_1+k_2)t}}{e^{(k_1+k_2)t}\int_{\alpha}^{\beta} p(x)\, i(x)\, dx} = \frac{p(x)\, i(x)}{\int_{\alpha}^{\beta} p(x)\, i(x)\, dx}$$

$$= f(x, 0) = f(x)$$

The needed adjustment for

$$\int_{\alpha}^{\beta} B(t-x)\, f(x)\, dx = B(t - T_{(t)}) \text{ is}$$

$$B^*(t - T_{(t)})\, e^{\frac{(k_1+k_2)}{2}\frac{\sigma^2}{T}} = \int_{\alpha}^{\beta} B(t-x)\, f(x)\, dx$$

$$= B(t - T_{(t)})$$

$$B(t) = R(0)\, e^{(k_1+k_2)t}\, B(t - T_0)\left[\frac{B^*(t - T_{(t)})}{B^*(t - T_0)}\right] e^{\frac{(k_1+k_2)}{2T_0}\sigma^2}$$

$$B(t) = R(0)\, e^{(k_1+k_2)t}\, B(t - T_0)\left[e^{-\frac{(k_1+k_2)\,\sigma_t^2}{4T_0} + \frac{(k_1+k_2)}{2T_0^2}\sigma^2 t^2 + \frac{(k_1+k_2)}{2T_0}\sigma^2}\right] \tag{I}$$

$$\log_e B(t) = \log_e R(0) + (k_1+k_2)\, t + \log_e B(t - T_0) - \frac{\sigma^2(k_1+k_2)^2}{4T_0}\, t$$
$$+ \frac{(k_1+k_2)^2}{2T_0^2}\sigma^2 t^2 + \frac{(k_1+k_2)}{2T_0}\sigma^2$$

putting $Y(t) = \log_e B(t)$ as before and taking $R(0) = 1$

$$Y(t) - Y(t - T_0) = (k_1+k_2)\, t - \frac{\sigma^2(k_1+k_2)^2\, t}{4T_0} + \frac{(k_1+k_2)^2}{2T_0^2}\sigma^2 t^2$$
$$+ \frac{(k_1+k_2)}{2T_0}\sigma^2$$

$$= a_0 + a_1 t + a_2 t^2$$

where

$$a_0 = \frac{\sigma^2(k_1+k_2)}{2T_0},\; a_1 = (k_1+k_2) - \frac{\sigma^2(k_1+k_2)^2}{4T_0} \text{ and } a_2 = \frac{\sigma^2(k_1+k_2)^2}{2T_0^2}$$

The solution of this difference equation is

$$B(t) = B(0)\, e^{\frac{(k_1+k_2)}{2} + \sigma^2 \frac{(k_1+k_2)}{2T_0}\left[\frac{1}{T_0} - \frac{(k_1+k_2)}{12}\right] t}$$
$$+ \frac{(k_1+k_2)}{2T_0}\left[1 + \frac{\sigma^2(k_1+k_2)}{4T_0}\right] t^2 + \frac{\sigma^2(k_1+k_2)^2}{6T_0^2}\, t^3 \tag{J}$$

APPENDIX B

We have

$$\int_{\alpha}^{\beta} p(x, t)\, i(x, t)\, dx = e^{(k_1+k_2)t} \int_{\alpha}^{\beta} p(x)\, i(x)\, dx = e^{(k_1+k_2)t} R_0 \qquad \text{(A)}$$

where R_0 is N.R.R. at $t = 0$

$$B(t) = e^{(k_1+k_2)t} \int_{\alpha}^{\beta} B(t - x)\, p(x)\, i(x)\, dx = e^{(k_1+k_2)t} B(t - T_{(t)}) \int_{\alpha}^{\beta} p(x)\, i(x)\, dx \qquad \text{(B)}$$

where $\alpha \leqslant T_{(t)} \leqslant \beta$

by the Mean Value theorem of integral calculus

$$\Rightarrow B(t) = R_0\, e^{(k_1+k_2)t} B(t - T_{(t)}) \qquad \text{(C)}$$

$B(t) = e^{(k_1+k_2)t} B(t - T_0)$ where $T_{(t)} = T_0$ the mean age of child bearing by Coale's first method and $R(0) = 1$.

$$\log_e B(t) = (k_1 + k_2)\, t + \log_e B(t - T_0)$$

Solving this simple equation by assuming an approximation solution, we get

$$B(t) = B(0)\, e^{\frac{(k_1 + k_2)}{2} t + \frac{(k_1 + k_2)}{2T_0} t^2} \qquad \text{(D)}$$

$$\begin{aligned} N(x, t) &= B(t - x)\, p(x, t) \\ &= B(0)\, e^{\frac{(k_1 + k_2)}{2} (t - x) + \frac{(k_1 + k_2)}{2T_0} (t - x)^2}\, p(x, t) \\ &= B(t)\, e^{-\frac{(k_1 + k_2)}{2} x + \frac{(k_1 + k_2)}{2T_0} x^2}\, e^{-\frac{(k_1 + k_2)}{T_0} tx}\, p(x, t) \qquad \text{(E)} \end{aligned}$$

Again $C(x, t)$ = The proportion of female population between age $[x, x + 1]$

$$\begin{aligned} C(x, t) &= \frac{B(t)\, e^{-\frac{(k_1 + k_2)}{2} x + \frac{(k_1 + k_2)}{2T_0} x^2}\, e^{-\frac{(k_1 + k_2)}{T_0} tx}\, p(x, t)}{N(t)} \\ &= b(t)\, e^{-\frac{(k_1 + k_2)}{2} x + \frac{(k_1 + k_2)}{2T_0} x^2}\, e^{-\frac{(k_1 + k_2)}{T_0} tx}\, p(x, t) \end{aligned}$$

where

$$b(t) = \frac{B(t)}{N(t)}$$

$$\Rightarrow C(x, t) = b(t)\, e^{-\frac{(k_1 + k_2)}{2} x + \frac{(k_1 + k_2)}{2T_0} x^2}\, e^{-r(t)x}\, p(x)\, e^{k_2 t}$$

where

$$\frac{k_1 + k_2)}{T_0}\, t = r$$

$$\Rightarrow \frac{C(x, t)}{C_s(x, t)} = \left[e^{k_2 t}\, \frac{b(t)}{b_s(t)} \right] e^{-\frac{(k_1 + k_2)}{2} x + \frac{(k_1 + k_2)}{2T_0} x^2} \qquad \text{(F)}$$

where $C_s(x, t) = b_s(t)\, e^{-r(t)x}\, e^{k_2 t}\, p(x)$